plotting the iterated function $\phi_2(x) = \phi(\phi(x))$; repeating the process for $\phi_2(x)$ instead of for $\phi(x)$, $y = \phi_4(x) = \phi\phi\phi\phi(x)$, and in the same way $y = \phi_k(x)$, may be plotted, when k is any power of 2. When k is not a power of 2, the construction of this paragraph generally leads to complicated figures.

A COURSE IN GEOMETRY FOR COLLEGE JUNIORS AND SENIORS.[1]

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Courses in geometry for college juniors and seniors are usually given under the title "Modern Geometry." In many instances, the scope of the course is restricted by designating it as a course either in "Modern Synthetic Geometry" or in "Modern Analytic Geometry." An examination of the catalogs of a number of the leading American universities and colleges shows that there is no cleancut agreement as to the content of a course under any of these three titles. The term "Modern Geometry" seems to have grown up in our mathematical nomenclature for the purpose of distinguishing a college course in geometry under this name from the more elementary course of the secondary school, this latter course being the *ancient* geometry of the Greeks practically unchanged. This ancient geometry was, as Professor Cajori expresses it in his history, decidedly special. In fact, one of its principal characteristics was a complete lack of general principles and methods. Courses in modern geometry do not differ from the courses in elementary geometry so much in their content as in the methods by which they produce not only the well-known results of ancient geometry but also many results which could not be obtained by the equipment and methods of ancient geometry.

The college junior and senior ready to begin a course in modern geometry has had the usual amount of this elementary algebra and geometry, the latter for the larger part, if not entirely, the geometry of the plane. He will also have had the usual amount of college algebra, trigonometry, analytic geometry and calculus with some applications to mechanics and geometry. He may in some instances have had courses in unified mathematics both in his preparatory work and in his freshman year in college, but the various subjects will in most cases have been taught within well-defined boundary lines with no attempt at generalization or coördination. As a rule nowhere in his course has the student had much work tending to emphasize the homogeneity of all his work in *geometry* and the general principles which underlie it in its entirety. It is the belief of the writer that the chief aim of the course for college juniors and seniors should be to remedy the defects of the elementary geometry mentioned above by supplying the generalizing methods and principles towards which modern thought has contributed so greatly. A brief historical review of the introduction of these new concepts will not be out of place here.

[1] Read before the Kansas Section of the Mathematical Association of America, March 18, 1916.

The introduction of infinitely great quantities into geometry by Kepler and the treatment of conics as projections of circles by Desargues and Pascal are among the first instances of the generalizing tendencies which characterize modern geometry. The announcement by Desargues and Pascal of the famous theorems named after them was the beginning of what is known as modern synthetic geometry. The introduction of the infinitely great into geometry by Kepler was closely coupled with the introduction of the infinitely small and led to the assumption by him of a powerful concept of modern geometry, namely, continuity. The middle of the seventeenth century witnessed the introduction into geometry by Descartes and others of the analytic method, another powerful agent in the new development. This new method retarded for a time the development of geometry on the purely synthetic side but more than compensated for this retardation by its generalizing tendencies and by the richness of its *suggestions* as to new concepts on the pure geometry side. The principle of duality was introduced from the geometric point of view by Gergonne early in the nineteenth century, was shortly afterwards applied to reciprocal polars by Poncelet, and was still further extended by Steiner. At about the same time Plücker introduced the principle of duality from the analytic point of view, coupling it with the idea of homogeneity. The introduction of the theory of imaginary points, lines and planes into projective geometry by Von Staudt followed soon after this. The concepts briefly outlined here form the principal instruments for the modern developments in geometry.

In many American universities the courses in modern geometry have been during the past two decades courses in advanced analytic geometry. The text followed was either Salmon's *Conic Sections*, Smith's *Conic Sections* or some other text mainly emphasizing the special properties of conic sections with some attention to homogeneous coördinates, abridged notation, etc., or else it was a text such as that of Miss Scott on *Modern Analytical Geometry*. During the past few years there has been a tendency evident in some institutions to swing the pendulum in the opposite direction and to make the course one in projective geometry from the synthetic point of view. But, as Professor Bussey points out in a strong plea for more synthetic work in geometry in No. 9, Vol. XX of THE AMERICAN MATHEMATICAL MONTHLY, a course in synthetic projective geometry does not commend itself to the prospective graduate student in mathematics at this stage of his development. For there is so much, even in geometry, that is of more importance to the student in his future work. The number of credit hours at the disposal of the student and the number and character of the students who are to take the course (and without them there will be no opportunity for the course) are matters that must also be considered in outlining a course, especially in the small college. In most of these institutions but one advanced course in geometry can be offered. The writer believes that this course should be both analytic and synthetic, emphasizing, if either, the synthetic side, as the analytic side has been in preponderance in the student's previous work. It should above all be an attempt to unify and coördinate the synthetic work of

the high school and the analytic work of the college as far as possible into a homogeneous whole. Such a course will be of benefit to the student whether he be a prospective high-school teacher of mathematics, a prospective research student in mathematics or one who is taking the course for its cultural value.

Thus, the concept of infinity and of parallel lines from the pure synthetic point of view will be greatly cleared up by an analytic discussion of the same, especially when homogeneous coördinates are used. The work in calculus now makes easy a comprehension of the idea of continuity and of limits in geometry which was not possible to the student of elementary geometry. The analytic treatment of imaginaries now makes possible a conception of the use and meaning of imaginary geometric elements. An illustration of this is the proof that a real straight line is the locus of the harmonic conjugates of a given point with respect to the two points in which a variable line through the point meets a conic, even when the variable line does not cut the conic in real points. The methods of the principle of duality when used synthetically appear much more valid to the student if they are reinforced by a thorough knowledge of the use of homogeneous coördinates and by a clear comprehension of the exact symmetry which arises in the analytic work. The beauty and the brevity of the synthetic work should, however, at all times be emphasized and the possibility of the avoidance of the analytic work, at times long and tedious, pointed out. The principle of central projection and the special cases arising from it can be studied simultaneously with work on linear transformations. When this is done a careful study may be made of the part played by the coefficients of the transformation. The geometry of each dimension above the first should be carefully built up on that of a lower dimension and coördinate systems in each dimension studied at the same time. It will then be possible to investigate the meaning and the properties of projective coördinates in each dimension and to point out the special systems, such as the metric Descartes system with its relationship to Euclidean geometry. The famous theorems of Desargues, Pascal, Brianchon, etc., should be carefully studied and the special theorems dependent on them investigated. Geometric addition and multiplication and their inverses ought to be explained and our ordinary algebraic operations shown as special cases. Quadrangular and other constructions may be illustrated by many examples. Throughout the course the idea of double ratio and its invariance under projection, the meaning of harmonics and their relation to involutions, the projective properties of conics, are all matters which should be emphasized.

Scarcity of time may make it necessary to make the treatment of some of the subjects mentioned in the previous paragraph very brief. The course is, however, believed by the writer to be such that the mind of the prospective teacher will be greatly enriched and that the prospective graduate student will have a first-class foundation for his future research work.